Hodge loci and absolute Hodge classes

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0 Introduction

Let $\pi : \mathcal{X} \to T$ be a family of smooth projective complex varieties. Assume $\mathcal{X}, \pi, T$ are defined over $\mathbb{Q}$. An immediate consequence of the fact that there are only countably many components of the relative Hilbert scheme for π, and that the relative Hilbert scheme (with fixed Hilbert polynomial) is defined over $\mathbb{Q}$, is the following: if the Hodge conjecture is true, the components of the Hodge locus in T are defined over $\overline{\mathbb{Q}}$, and their Galois transforms are again components of the Hodge locus. (We recall later on the definition of the components of the Hodge locus.) In [4], it is proven that the components of the Hodge locus (and even the components of the locus of Hodge classes, which is a stronger notion) are algebraic sets, while Hodge theory would give them only a local structure of closed analytic subsets (see [10], 5.3.1).

In this paper, we give simple sufficient conditions for components of the Hodge locus to be defined over $\overline{\mathbb{Q}}$ (and their Galois transforms to be also components of the Hodge locus). This criterion of course does not hold in full generality, and it particular does not say anything about the definition field of an isolated point in the Hodge locus. But in practice, it is reasonably easy to check and allows to conclude in some explicit cases, where the Hodge conjecture is not known to hold. We give a few examples of applications in section 3.

We will first relate this geometric language to the notion of absolute Hodge classes (as we only deal with the de Rham version, we will not use the terminology of Hodge cycles of [5]), and explain why this notion allows to reduce the Hodge conjecture to the case of varieties defined over $\overline{\mathbb{Q}}$, thus clarifying a question asked to us by V. Maillot and Ch. Soulé.

Let us recall the notion of (de Rham) absolute Hodge class (cf [5]). Let X^{an} be a complex projective manifold and $\alpha \in Hdg^{2k}(X^{an})$ be a rational Hodge class. Thus α is rational and

$$\alpha \in F^k H^{2k}(X^{an}, \mathbb{C}) \cong \mathbb{H}^{2k}(X^{an}, \Omega^{\bullet \geq k}_{X^{an}}). \tag{0.1}$$

Here, the left hand side is Betti cohomology of the complex manifold X^{an} and the isomorphism of (0.1) is induced by the resolution

$$0 \to \mathbb{C} \stackrel{(2i\pi)^k}{\to} \mathcal{O} \stackrel{d}{\to} \Omega_X \to \ldots \to \Omega_X^n \to 0,\ n = dim\, X$$

of the constant sheaf $\mathbb{C}$ on X^{an}. The right hand side in (0.1) can be computed, by GAGA principle, as the hypercohomology of the algebraic variety X with value in the complex of algebraic differentials:

$$\mathbb{H}^{2k}(X^{an},\Omega_{X^{an}}^{\bullet\geq k})\cong\mathbb{H}^{2k}(X,\Omega_X^{\bullet\geq k}).$$

Let us denote by $\mathcal{E}$ the set of fields embeddings of $\mathbb{C}$ in $\mathbb{C}$. For each element σ of $\mathcal{E}$, we get a new algebraic variety X_σ defined over $\mathbb{C}$, and we have a similar isomorphism for X_σ. Thus the class α provides a (de Rham or Betti) complex cohomology class

$$\alpha_\sigma\in\mathbb{H}^{2k}(X_\sigma,\Omega_{X_\sigma}^{\bullet\geq k})=F^kH^{2k}(X_\sigma^{an},\mathbb{C})$$

for each $\sigma\in\mathcal{E}$.

Definition 0.1 *(cf [5])* The class α is said to be (de Rham) absolute Hodge if α_σ is a rational cohomology class for each σ.

We will introduce in section 1 the notion of weakly absolute Hodge class. In the definition above, we ask that each α_σ is proportional to a rational cohomology class.

We first prove in this Note the following statement, which answers a question asked by Vincent Maillot and Christophe Soulé:

Proposition 0.2 *Assume the Hodge conjecture is known for varieties $X_{\overline{\mathbb{Q}}}$ defined over $\overline{\mathbb{Q}}$ and (weakly) absolute Hodge classes α on them. Then the Hodge conjecture is true for (weakly) absolute Hodge classes.*

Remark 0.3 It is easy to see, (see Lemma 1.4) that a weakly absolute Hodge class α on a variety defined over $X_{\overline{\mathbb{Q}}}$ is defined over $\overline{\mathbb{Q}}$, that is $\alpha\in\mathbb{H}^{2k}(X_{\overline{\mathbb{Q}}},\Omega_{X_{\overline{\mathbb{Q}}}}^{\bullet\geq k})$.

Remark 0.4 In the statement of the Proposition, we fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, and so α determines a class in $\mathbb{H}^{2k}(X_{\mathbb{C}},\Omega_{X_\sigma}^{\bullet\geq k})=F^kH^{2k}(X_{\mathbb{C}}^{an},\mathbb{C})$, which is assumed to be rational. If the Hodge conjecture is true for this class, then for any other embedding σ of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, the class α_σ is also rational, and the Hodge conjecture is also true for this Hodge class. Thus the statement makes sense and is independent of the choice of embedding.

We next turn to the problem of whether Hodge classes should be absolute. Let X be a complex projective manifold, with a deformation family $\pi:\mathcal{X}\to T$ defined over $\mathbb{Q}$, that is X is a fiber X_0, for some complex point $0\in T(\mathbb{C})$, and let $\alpha\in Hdg^{2k}(X)$ be a primitive Hodge class. We show the following:

Theorem 0.5 *1) Assume that for one irreducible component S passing through α of the locus of Hodge classes, there is no constant sub-variation of Hodge structure of $R^{2k}\pi_{S*}\mathbb{Q}_{prim}$ on S, except for $\mathbb{Q}\alpha_t$. Then α is weakly absolute.*

2) Let us weaken the assumptions on S by asking that any constant sub-variation of Hodge structure of $R^{2k}\pi_{S*}\mathbb{Q}_{prim}$ on S is of type (k,k). Then, $p(S_{red})$ is defined over $\overline{\mathbb{Q}}$, and satisfies the property that its Galois translates are also of the form $p(S'_{red})$ for some irreducible component S' of the locus of Hodge classes.

Here $\pi_S:\mathcal{X}_S\to S_{red}$ is obtained by base change $p:S_{red}\to T$. In statement 2), the Hodge locus of α is defined as the projection to T (via the projection map p) of the connected component of the locus of Hodge classes passing through α. We will describe in section 1 their natural schematic structure.

Statement 1) will imply, by Lemma 1.4 proven it next section, that under the same assumptions, the Hodge locus of α is defined over $\overline{\mathbb{Q}}$ and its image under any element of $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$ is again a component of the Hodge locus.

An immediate Corollary of Theorem 0.5,1) is the following simple statement :

Corollary 0.6 *Assume that the infinitesimal Torelli theorem holds for the variation of Hodge structure on $R^{2k}\pi_*\mathbb{Q}_{prim}$. Assume that one component S passing through α of the locus of Hodge classes has positive dimension, and that the only proper non trivial sub-Hodge structure of $H^{2k}(X,\mathbb{Q})_{prim}$ is $\mathbb{Q}\alpha$. Then α is weakly absolute.*

Note that the assumption that S has positive dimension is satisfied once $h^{k-1,k+1} := rk\,H^{k-1,k+1}(X)_{prim} < dim\,T$ (cf [10], Proposition 5.14).

Proof. Indeed, a constant sub-variation of Hodge structure of $R^{2k}\pi_{S*}\mathbb{Q}_{prim}$ on S must then be (by taking the fiber at the point 0 corresponding to X) either equal to $R^{2k}\pi_{S*}\mathbb{Q}_{prim}$ or to $\mathbb{Q}\alpha$. The first case is impossible by the Torelli assumption, and $dim\,S > 0$. Thus the assumptions of Theorem 0.5,1) are satisfied. ∎

Case 2) of Theorem 0.5 leads to the following generalization of Proposition 0.2:

Proposition 0.7 *Suppose the Hodge conjecture is true for Hodge classes on smooth projective varieties defined over $\overline{\mathbb{Q}}$. Then under the assumptions of Theorem 0.5, 2), the class α is algebraic.*

Section 1 is devoted to the discussion of absolute and weakly absolute Hodge classes in terms of the corresponding components of the locus of Hodge classes and components of the Hodge locus.

In section 2, we prove the results stated in this introduction.

We give in the last section variants and applications of Theorem 0.5. In Theorem 3.1, we give an algebraic (Zariski open) criterion on a Hodge class $\alpha \in F^kH^{2k}$ in order that the assumptions of Theorem 0.5 are satisfied at least at a general point of the connected component $\widetilde{S}_\alpha$ of the locus of Hodge classes passing through α. Of course, except in level 2, where we can use the Green density criterion, it is hard to decide if there are many Hodge classes in the Zariski open set of F^kH^{2k} where this criterion is satisfied. We give examples in level 2, where this criterion is satisfied in a Zariski dense open set, in which there are "many" Hodge classes. In one of these examples, the Hodge conjecture is not known to hold for these classes.

The second application (Theorem 3.5) concerns the period map. Under a reasonable assumption on the infinitesimal variation of Hodge structures on the primitive cohomology of the fibers of a family $\pi : \mathcal{X} \to T$ of projective varieties defined over $\mathbb{Q}$, we conclude that any component W dominating T by the first projection of the set of pairs $(t,t') \in T \times T$ such that the Hodge structures on $H^n(X_t,\mathbb{Q})_{prim}$ and $H^n(X_{t'},\mathbb{Q})_{prim}$ are isomorphic, is defined over $\overline{\mathbb{Q}}$.

1 Absolute and weakly absolute Hodge classes

Let us introduce the following variant of the notion of absolute Hodge class.

Definition 1.1 The class α is said to be weakly (de Rham) absolute Hodge if for each $\sigma \in \mathcal{E}$, α_σ is a multiple $\lambda_\sigma\gamma_\sigma$, where $\gamma_\sigma \in H^{2k}(X^{an},\mathbb{Q})$ is a rational cohomology class (hence a Hodge class) and $\lambda_\sigma \in \overline{\mathbb{Q}}$.

Remark 1.2 It turns out that the condition $\lambda_\sigma \in \overline{\mathbb{Q}}$ is automatically satisfied. Indeed, consider the primitive decomposition of α with respect to the polarization given by the projective embedding of X.

$$\alpha = \sum_{2k-2r\geq 0, 2r\leq n} c_1(L)^{k-r}\alpha_r,\ n = dim\, X,$$

where $\alpha_r \in H^{2r}(X, \mathbb{Q})_{prim}$.

Then the primitive decomposition of α_σ is given by

$$\alpha_\sigma = \sum_{2k-2r\geq 0, 2r\leq n} c_1(L_\sigma)^{k-r}\alpha_{r,\sigma},$$

and thus, if $\alpha_\sigma = \lambda_\sigma \gamma_\sigma$, with $\gamma_\sigma \in H^{2k}(X_\sigma^{an}, \mathbb{Q})$, then for each r, we get

$$\alpha_{r,\sigma} = \lambda_\sigma \gamma_{\sigma,r}, \tag{1.1}$$

where $\gamma_{\sigma,r}$ is the degree $2r$ primitive component of γ_σ, and thus is a rational cohomology class. But we know by the second Hodge-Riemann bilinear relations that if $\alpha_r \neq 0$, we have $\int_X c_1(L)^{n-2r}\alpha_r^2 \neq 0$. This is a rational number, which is also equal to $\int_{X_\sigma} c_1(L_\sigma)^{n-2r}\alpha_{r,\sigma}^2$. On the other hand, as $\gamma_{\sigma,r}$ is a rational cohomology class, we also have $\int_{X_\sigma} c_1(L_\sigma)^{n-2r}\gamma_{\sigma,r}^2 \in \mathbb{Q}$, and thus, from the equalities

$$\int_X c_1(L)^{n-2r}\alpha_r^2 = \int_{X_\sigma} c_1(L_\sigma)^{n-2r}\alpha_{r,\sigma}^2 = \lambda_\sigma^2 \int_{X_\sigma} c_1(L_\sigma)^{n-2r}\gamma_{\sigma,r}^2, \tag{1.2}$$

we get $\lambda_\sigma^2 \in \mathbb{Q}$.

Geometrically, the meaning of these notions is the following (see also [7]): Let $\pi : \mathcal{X} \to T$ be a family of deformations of X, which is defined over $\mathbb{Q}$ (here T is not supposed to be geometrically irreducible, and thus the assumption is not restrictive on X). There is then the algebraic vector bundle F^kH^{2k} on T, defined over $\mathbb{Q}$, which is the total space of the locally free sheaf $F^k\mathcal{H}^{2k} = R^{2k}\pi_*\Omega_{\mathcal{X}/T}^{\bullet\geq k}$ on T. We will use the following terminology (see [4]). The *locus of Hodge classes* for the family above, in degree $2k$ is the set of pairs $(X_t, \alpha_t) \in F^kH^{2k}(\mathbb{C})$, such that $\alpha_t \in H^{2k}(X_t^{an}, \mathbb{C})$ is rational (hence a Hodge class).

The components of the *Hodge locus* are the image in T, via the natural projection $p : F^kH^{2k} \to T$, of the connected components of the locus of Hodge classes. If α is a Hodge class on X, the *Hodge locus of* α is the image in T of the connected component of the locus of Hodge classes passing through α.

Notice that the locus of Hodge classes is obviously locally a countable union of closed analytic subsets in $F^kH^{2k}(\mathbb{C})$ Indeed, if $\alpha \in F^kH^{2k}(X_t^{an}, \mathbb{C}) \cap H^{2k}(X^{an}, \mathbb{Q})$, then in a simply connected neighbourhood U of $t \in T$, we have a trivialization of the locally constant sheaf $R^{2k}\pi_*^{an}\mathbb{C}$, which induces a trivialization of the corresponding vector bundle H^{2k} and gives a composed holomorphic map:

$$\psi : F^kH^{2k} \hookrightarrow H^{2k} \to H^{2k}(X_t, \mathbb{C}), \tag{1.3}$$

where H^{2k} is the total space of the locally free sheaf

$$\mathcal{H}^{2k} = R^{2k}\pi_*\Omega_{\mathcal{X}/T}^\bullet = R^{2k}\pi_*^{an}\mathbb{C} \otimes \mathcal{O}_T$$

on T.

Then, over U, the locus of Hodge classes identifies to $\psi^{-1}(H^{2k}(X_t,\mathbb{Q}))$, which is a countable union of fibers of ψ. This defines a natural schematic structure on the connected components of the locus of Hodge classes.

Similarly, the local description of the Hodge locus of α is as follows: we can locally extend α to a locally constant section $\tilde{\alpha}$ of $R^{2k}\pi_*^{an}\mathbb{Q}$. Then $\tilde{\alpha}$ gives in particular a holomorphic section of the vector bundle $\mathcal{H}^{2k} := R^{2k}\pi_*^{an}\mathbb{C}\otimes\mathcal{O}_T$. Then the Hodge locus of α is simply defined by the condition

$$\tilde{\alpha}^{\leq k-1} = 0,$$

where $\tilde{\alpha}^{\leq k-1}$ is the projection of $\tilde{\alpha}$ in the quotient $\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$. This again defines the schematic structure of the Hodge locus of α.

It is clear from these descriptions that the projection from the locus of Hodge classes to the Hodge locus is a local immersion which is open onto an union of local analytic branch of the Hodge locus.

Cattani, Deligne and Kaplan proved in fact the following much stronger result concerning the structure of the locus of Hodge classes (cf [4]):

Theorem 1.3 *The connected components of the locus of Hodge classes are algebraic subsets of the algebraic vector bundle F^kH^{2k}.*

The conjecture that Hodge classes are absolute Hodge is equivalent to saying that the locus of Hodge classes is a countable union of algebraic subsets defined over $\mathbb{Q}$. To see this, note that a given Hodge class is absolute Hodge if and only if its $\mathbb{Q}$-Zariski closure in F^kH^{2k} is contained in the locus of Hodge classes. It is then clear by the Noetherian property that for countably many generically chosen Hodge classes α_i, the locus of Hodge classes must be equal to the union of the $\mathbb{Q}$-Zariski closures of the α_i's.

The statement that Hodge classes are weakly absolute Hodge implies the facts that the locus of Hodge classes is a countable union of algebraic subsets of F^kH^{2k} defined over $\overline{\mathbb{Q}}$ and that the Hodge locus is a countable union of algebraic subsets of T defined over $\mathbb{Q}$. More precisely, we have :

Lemma 1.4 *Let $\alpha\in H^{2k}(X^{an},\mathbb{Q})$ be a weakly absolute Hodge class. Then the connected component $\widetilde{S}_\alpha$ of the locus of Hodge classes passing through α is defined (schematically) over $\overline{\mathbb{Q}}$, and so is the Hodge locus of α. Furthermore the Galois images of the Hodge locus of α are also (schematically) components of the Hodge locus.*

Proof. We know by Theorem 1.3 that $\widetilde{S}_\alpha$ is algebraic, and it is by definition connected. We make the base change $\widetilde{S}_{\alpha,red}\to T$, where we replace if necessary $\widetilde{S}_\alpha$ by a Zariski open set, in order to make the reduced scheme $\widetilde{S}_{\alpha,red}$ smooth. Then the corresponding family

$$\pi_\alpha : \mathcal{X}_\alpha \to \widetilde{S}_{\alpha,red}$$

admits the locally constant section $\tilde{\alpha}\in H^0(\widetilde{S}_{\alpha,red}, R^{2k}\pi_{\alpha*}\mathbb{Q})$. By the global invariant cycle theorem [6], there exists a class $\beta\in H^{2k}(\overline{\mathcal{X}}_\alpha,\mathbb{Q})$ which is of type (k,k) and restricts to $\tilde{\alpha}_t$ on each fiber X_t of the family $\mathcal{X}_\alpha$. In fact, we can even make this class canonically defined by choosing an ample line bundle $\mathcal{L}$ on $\overline{\mathcal{X}}_\alpha$, which allows to define a polarization on $H^{2k}(\overline{\mathcal{X}}_\alpha, \mathbb{Q})$ (see also the proof of Proposition 0.2 or [2] for more details). Then β is canonically defined if we impose that β lies in the orthogonal complement of $Ker\, rest_X$ with respect to this polarization.

Let now α be weakly absolute. Then the class β_σ on $\mathcal{X}_{\alpha,\sigma}$ restricts to $\alpha_\sigma = \lambda_\sigma \gamma_\sigma$ on X_σ, where γ_σ is rational, and is in the orthogonal complement of $Ker\, rest_{X_\sigma}$ with respect to the polarization induced by $\mathcal{L}_\sigma$. It thus follows that $\frac{1}{\lambda_\sigma}\beta_\sigma$, which restricts to γ_σ, has to be rational (hence is a Hodge class). Let $\tilde{\gamma}$ be the locally constant section of $R^{2k}\pi_{\alpha*}\mathbb{Q}$ on $\widetilde{S}_{\alpha,red}$ obtained by restricting $\frac{1}{\lambda_\sigma}\beta_\sigma$. We conclude that we have an inclusion

$$\frac{1}{\lambda_\sigma}\sigma(\widetilde{S}_{\alpha,red}) \subset \widetilde{S}_{\gamma_\sigma,red}, \tag{1.4}$$

which is easily checked to extend in fact to a schematic identification

$$\frac{1}{\lambda_\sigma}\sigma(\widetilde{S}_\alpha) = \widetilde{S}_{\gamma_\sigma}. \tag{1.5}$$

Indeed, this follows from the flatness of the sections $\tilde{\alpha}_\sigma$ and $\tilde{\gamma}_\sigma$, from the fact that λ_σ has to be constant along $\sigma(\widetilde{S}_{\alpha,red})$ by formula (1.2), and from the fact that $\widetilde{S}_{\gamma_\sigma}$ is by definition connected.

As

$$p(\frac{1}{\lambda}\sigma(\widetilde{S}_\alpha)) = p(\sigma(\widetilde{S}_\alpha))) = \sigma(p(\widetilde{S}_\alpha)),$$

we conclude from (1.5) that the image via σ of the Hodge locus of α is also a component of the Hodge locus.

Finally, to see that if α is weakly absolute Hodge, then $\widetilde{S}_\alpha \subset F^k H^{2k}$ is defined over $\overline{\mathbb{Q}}$, we use equality (1.5), applied to $\sigma \in \mathcal{E}$ together with the fact noticed in Remark 1.2 that $\lambda_\sigma^2 \in \mathbb{Q}$. It follows that the constant $\lambda_\sigma \in \overline{\mathbb{Q}}$ can take only countably many values, and in particular, there are only countably many Galois transforms $\sigma(\widetilde{S}_\alpha)$, and as we know that $\widetilde{S}_\alpha$ is algebraic, this implies that $\widetilde{S}_\alpha$ is defined over $\overline{\mathbb{Q}}$. ∎

2 Proof of Theorem 0.5 and Propositions 0.2, 0.7.

Proof of Proposition 0.2. Let (X^{an}, α) be a pair consisting of a projective complex manifold and an absolute (resp. a weakly absolute) rational Hodge class. By the geometric interpretation given above, and by Lemma 1.4 in the weakly absolute case, it follows that there exist smooth irreducible quasi-projective varieties $\mathcal{X}$, T defined over $\overline{\mathbb{Q}}$, a projective morphism $\pi : \mathcal{X} \to T$, and a locally constant global section

$$\tilde{\alpha} \in H^0(T, R^{2k}\pi_*\mathbb{Q}),$$

such that X is one fiber of π and α is the restriction of $\tilde{\alpha}$ to this fiber.

Deligne's global invariant cycle theorem [6] says now that for any smooth completion $\overline{\mathcal{X}}$ of $\mathcal{X}$, there exists a Hodge class $\beta \in Hdg^{2k}(\overline{\mathcal{X}})$ such that

$$\beta_{|X} = \alpha.$$

Of course, we may also choose $\overline{\mathcal{X}}$ defined over $\overline{\mathbb{Q}}$. In order to conclude, we claim that we may choose β to be absolute Hodge (resp. weakly absolute Hodge). Indeed, we will deduce from this, under the assumptions of Proposition 0.2, that β is the class of an algebraic cycle, and then, so is its restriction α.

To prove the claim, consider the morphism of rational Hodge structures

$$H^{2k}(\overline{\mathcal{X}}^{an}, \mathbb{Q}) \to H^{2k}(X^{an}, \mathbb{Q}).$$

The left hand side can be polarized using a ample line bundle $\mathcal{L}$ on $\overline{\mathcal{X}}$. (That is, we use the Lefschetz decomposition with respect to this polarization, and change the signs of the natural intersection pairing

$$(\alpha_r, \beta_r) = \int_{\overline{\mathcal{X}}} c_1(\mathcal{L})^{N-2r} \alpha_r \cup \beta_r,\ N = dim\, \mathcal{X}$$

on the pieces of the Lefschetz decomposition with r even, in order to get a polarized Hodge structure.) Thus we conclude that there is an orthogonal decomposition

$$H^{2k}(\overline{\mathcal{X}}^{an}, \mathbb{Q}) = A \oplus B$$

into the sum of two Hodge structures, where the first one identifies via restriction to its image in $H^{2k}(X, \mathbb{Q})$, while the second one is the kernel of the restriction map. B is a sub-Hodge structure of $H^{2k}(\overline{\mathcal{X}}^{an}, \mathbb{Q})$ and A is then defined as the orthogonal of B under the metric described above on $H^{2k}(\overline{\mathcal{X}}^{an}, \mathbb{Q})$.

We define then β to be the unique element of A which restricts to α.

For each element σ of $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$, we get a line bundle $\mathcal{L}_\sigma$ on $\overline{\mathcal{X}}^{an}_\sigma$, a sub-Hodge structure $B_\sigma := Ker\, rest_{X_\sigma}$, and the isomorphism

$$H^{2k}(\overline{\mathcal{X}}^{an}, \mathbb{C}) \cong H^{2k}(\overline{\mathcal{X}}^{an}_\sigma, \mathbb{C}) \tag{2.1}$$

commutes with restrictions maps and is compatible with the polarizations given by $\mathcal{L}$ and $\mathcal{L}_\sigma$. Thus we get similarly a rational sub-Hodge structure A_σ of $H^{2k}(\overline{\mathcal{X}}^{an}_\sigma, \mathbb{Q})$ and there is a commutative diagram where the horizontal maps are restrictions maps and thus are defined on rational cohomology, and the vertical maps are induced by the comparison isomorphism (2.1) :

$$\begin{array}{ccc} A \otimes \mathbb{C} & \hookrightarrow & H^{2k}(X, \mathbb{C}) \\ \| & & \| \\ A_\sigma \otimes \mathbb{C} & \hookrightarrow & H^{2k}(X_\sigma, \mathbb{C}) \end{array}.$$

It follows from this that if α is absolute Hodge (resp. weakly absolute Hodge), so is β.

∎

Proof of Theorem 0.5. 1) Let (X, α) be as in the statement of the Theorem. By Theorem 1.3, the component passing through α of the locus of Hodge classes is an algebraic set. Let S be an irreducible component of this set containing (X, α), and satisfying the assumption of Theorem 0.5,1). Replacing S by a Zariski open set of S_{red}, we may assume that S is smooth. There is by base change a projective family $\pi_S : \mathcal{X}_S \to S$ together with a tautological flat section

$$\tilde{\alpha} \in H^0(S, R^{2k}\pi_{S*}\Omega^{\bullet \geq k}_{\mathcal{X}_S/S}),$$

with value α_t at each t.

Let $\overline{\mathcal{X}}_S$ be a smooth completion of $\mathcal{X}_S$. The global invariant cycle theorem says that there exists a class $\beta \in H^{2k}(\overline{\mathcal{X}}_S, \mathbb{Q}) \cap F^k H^{2k}(\overline{\mathcal{X}}_S, \mathbb{Q})$ such that $\beta_{|X} = \alpha$. On the other hand, the vector space

$$H^{2k}(\overline{\mathcal{X}}_S, \mathbb{Q})_{|X_t} \cap H^{2k}(X_t, \mathbb{Q})_{prim}$$

is a constant sub-Hodge structure of $H^{2k}(X, \mathbb{Q})_{prim}$. Thus, by our assumption on S, we conclude that it must be equal to $\mathbb{Q}\alpha_t$. It follows that the complex vector space

$$\mathbb{H}^{2k}(\overline{\mathcal{X}}_S, \Omega^\bullet_{\overline{\mathcal{X}}_S})_{|X_t} \cap \mathbb{H}^{2k}(X_t, \Omega^\bullet_{X_t})_{prim}$$

has rank 1 and is generated by α.

Let $\sigma \in \mathcal{E}$. We want to show that the class

$$\alpha_\sigma \in \mathbb{H}^{2k}(X_\sigma, \Omega^{\bullet \geq k}_{X_\sigma}) \subset H^{2k}(X^{an}_\sigma, \mathbb{C})$$

is of the form $\lambda_\sigma \gamma_\sigma$, where γ_σ is rational.

But σ provides a new family $\overline{\mathcal{X}}_{S,\sigma}$ fibered over S_σ with fiber $X_{t,\sigma}$, such that the vector space

$$\mathbb{H}^{2k}(\overline{\mathcal{X}}_{S,\sigma}, \Omega^\bullet_{\overline{\mathcal{X}}_{S,\sigma}})_{|X_\sigma} \cap \mathbb{H}^{2k}(X_\sigma, \Omega^\bullet_{X_\sigma})_{prim} \tag{2.2}$$

has rank 1 and is generated by α_σ. It follows that the intersection of the image of the restriction map

$$H^{2k}(\overline{\mathcal{X}}^{an}_{S,\sigma}, \mathbb{Q}) \to H^{2k}(X^{an}_\sigma, \mathbb{Q}), \tag{2.3}$$

with $H^{2k}(X^{an}_\sigma, \mathbb{Q})_{prim}$ has rank 1.

Thus we have $\alpha_\sigma = \lambda_\sigma \gamma_\sigma$ for some rational primitive Hodge class γ_σ on X_σ, and some non zero complex coefficient λ_σ. By Remark 1.2, we have $\lambda_\sigma \in \overline{\mathbb{Q}}$, and thus α is weakly absolute.

2) The proof of 2) is very similar. Indeed, with the same notations as above, we find that α belongs to the sub-Hodge structure

$$H^{2k}(\overline{\mathcal{X}}^{an}_S, \mathbb{Q})_{|X^{an}} \cap H^{2k}(X^{an}, \mathbb{Q})_{prim},$$

which is the fiber at 0 of the locally constant sub-Hodge structure

$$H^{2k}(\overline{\mathcal{X}}^{an}_S, \mathbb{Q})_{|X^{an}_t} \cap H^{2k}(X^{an}_t, \mathbb{Q})_{prim},\ t \in S,$$

hence must be a trivial sub-Hodge structure. This assumption is algebraic, as it can be translated into the fact that the vector space

$$\mathbb{H}^{2k}(\overline{\mathcal{X}}_S, \Omega^\bullet_{\overline{\mathcal{X}}_S})_{|X} \cap \mathbb{H}^{2k}(X, \Omega^\bullet_X)_{prim}$$

is equal to

$$\mathbb{H}^{2k}(\overline{\mathcal{X}}_S, \Omega^{\bullet \geq k}_{\overline{\mathcal{X}}_S})_{|X} \cap \mathbb{H}^{2k}(X, \Omega^{\bullet \geq k}_X)_{prim}.$$

Let now $\sigma \in \mathcal{E}$. We conclude from the above that the sub-Hodge structure

$$H^{2k}(\overline{\mathcal{X}}^{an}_{S,\sigma}, \mathbb{Q})_{|X^{an}_\sigma} \cap H^{2k}(X^{an}_\sigma, \mathbb{Q})_{prim},$$

to which α_σ belongs, is trivial. Thus we can write $\alpha_\sigma = \sum_{i=1}^{i=N} \alpha_i\gamma_i$, where γ_i are independent rational Hodge classes on X_σ^{an} coming from $\overline{\mathcal{X}}_{S,\sigma}^{an}$ and the λ_i are complex coefficients. As α_σ gives a flat section of the bundle $F^k\mathcal{H}^{2k}$ on $\sigma(S)$, and the γ_i are locally constant on $\sigma(S_{red})$, we conclude that the λ_i's are constant on $\sigma(S_{red})$. We claim that for a generic choice of *rational* coefficients λ'_i, $1 \leq i \leq N$, $\sigma(p(S_{red}))$ is equal to $p(S''_{red})$ where S'' is an irreducible component of the locus of Hodge classes passing through $\sum_{i=1}^{i=N} \lambda'_i\gamma_i$.

Assuming the claim, this shows that there are only countably many Galois transforms of $p(S_{red})$ and thus, because $p(S_{red})$ is algebraic, this implies that $p(S_{red})$ is defined over $\overline{\mathbb{Q}}$. The claim also gives the second part of the statement.

To prove the claim, we choose a simply connected neighborhood U of the point $\sigma(0) \in T(\mathbb{C})$. Over U, we can consider the map $\psi : F^kH^{2k}_{|U} \to H^{2k}(X_\sigma^{an}, \mathbb{C})$ of (1.3). Then for any choice of complex coefficients μ_i, we know that $p(\psi^{-1}(\sum_i \mu_i\gamma_i))$ contains $p(\sigma(S_{red})) \cap U$, and that for $(\mu_1, \ldots, \mu_N) = (\lambda_1, \ldots, \lambda_N)$, $p(\sigma(S_{red})) \cap U$ is the reduction of an irreducible component of $p(\psi^{-1}(\sum_i \mu_i\gamma_i))$. By lower semi-continuity of the dimension of the fibers of ψ, we conclude that the later property remains true for $(\mu_i) \in \mathbb{C}^N$ in a Zariski open set of coefficients, and thus in particular for some N-uple $(\lambda'_i) \in \mathbb{Q}^N$.

Having this, we proved that for some irreducible analytic component S' of $\psi^{-1}(\sum_{i=1}^{i=N} \lambda'_i\gamma_i)$, the two analytic subsets $\sigma(p(S_{red})) \cap U$ and $p(S'_{red})$ of U coincide. Because $\sigma(p(S_{red}))$ is irreducible and reduced, and because by Theorem 1.3, $\psi^{-1}(\sum_{i=1}^{i=N} \lambda'_i\gamma_i)$ is an open set in an irreducible algebraic subset S'' of F^kH^{2k}, (an irreducible component of a connected component of the locus of Hodge classes), we get by analytic continuation that $\sigma(p(S_{red})) = p(S''_{red})$.

■

Remark 2.1 *The schematic structure of the locus where a combination $\sum_i \mu_i\gamma_i$ remains in F^kH^{2k} may depend on the μ_i, even if we know that the corresponding reduced algebraic set does not depend generically on the μ_i's. This is why we have to restrict here to the reduced subschemes.*

Let us conclude this section by giving the proof of proposition 0.7.

Proof of Proposition 0.7. Indeed, with the same notations as above, we just proved that $S' := p(S_{red})$ is defined over $\overline{\mathbb{Q}}$. We also know that the only locally constant sub-Hodge structure of $R^{2k}\pi_*\mathbb{Q}_{prim}$ is of type (k, k). As monodromy acts in a finite way on the set of Hodge classes of X_t, $t \in S'$ generic, there is an étale cover S'' of the smooth part of S', also defined over $\overline{\mathbb{Q}}$, on which this monodromy action becomes trivial. Thus we have by base change a family $\pi'' : \mathcal{X}_{S''} \to S''$, together with a global section $\tilde{\alpha}$ of $R^{2k}\pi''_*\mathbb{Q}_{prim}$, whose restriction to X_0 is equal to α. The global invariant cycle theorem now says that there exists a Hodge class β on a smooth compactification $\overline{\mathcal{X}}_{S''}$, which we may assume defined over $\overline{\mathbb{Q}}$, restricting to α. If the Hodge conjecture is true for Hodge classes on varieties defined over $\overline{\mathbb{Q}}$, it is then true for β and thus also for α. ■

3 Variants and applications

Let us give to start with an infinitesimal criterion which will guarantee that the assumptions of Theorem 0.5, 1) are satisfied by an irreducible component of $\widetilde{S}_\alpha$. This will then give as a consequence of Theorem 0.5 an algebraic criterion (Theorem 3.1) for a Hodge class $\alpha \in F^k H^{2k}(\mathbb{C})$, to be weakly absolute.

We assume again that $\pi : \mathcal{X} \to T$ is a family of projective varieties defined over $\mathbb{Q}$, and we denote as before by $F^k H^{2k}$ the algebraic vector bundle whose sheaf of sections is $(R^{2k}\pi_*\Omega^{\bullet \geq k}_{\mathcal{X}/T})_{prim}$, which admits as a quotient the bundle $H^{k,k}$ whose sheaf of sections is $(R^k\pi_*\Omega^k_{\mathcal{X}/T})_{prim}$. This is an algebraic vector bundle defined over $\mathbb{Q}$. We have the $\mathcal{O}_T$-linear map which describes the infinitesimal variation of Hodge structure

$$\overline{\nabla} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega_T,$$

which is defined using the Gauss-Manin connection and Griffiths transversality (cf [10], 5.1.2). Here $\mathcal{H}^{p,q} := (R^q\pi_*\Omega^p_{\mathcal{X}/T})_{prim}$.

The assumption of positive dimension for the Hodge loci is automatically satisfied if $h^{k-1,k+1}(X)_{prim} < dim\, T$. This is proved in [10], where it is shown that the Hodge loci in T for the variation of Hodge structure on $H^{k-1,k+1}(X_t)_{prim}$ can be defined by at most $h^{k-1,k+1}(X)_{prim} < dim\, T$. We assume below that T is smooth.

Let $\alpha \in H^{2k}(X,\mathbb{Q})_{prim}$ be a Hodge class, where $X = X_0$ is a fiber of π, $0 \in T(\mathbb{C})$. Let $\lambda \in H^{k,k}$ be the projection of $\alpha \in F^k H^{2k}$ in $H^{k,k}$.

Let us assume that the map

$$\mu : T_{T,0} \to H^{k-1,k+1}(X_0)$$

given by $\mu(v) = \overline{\nabla}_v(\lambda)$ is surjective. Let K_λ be its kernel. K_λ is the tangent space of the Hodge locus of α at 0 (cf [10]???). We have the following algebraic criterion on λ, for α to be weakly absolute:

Theorem 3.1 *Assume that*

1) μ_λ is surjective.

2) For $p > k$, $p + q = 2k$ the map

$$\overline{\nabla}_0 : H^{p,q}(X_0)_{prim} \to H^{p-1,q+1}(X_0) \otimes K^*_\lambda,$$

obtained by restriction of $\overline{\nabla}$, is injective.

3) The map

$$H^{k,k}(X_0)_{prim} \to H^{k-1,k+1}(X_0) \otimes K^*_\lambda,$$

obtained by restriction of $\overline{\nabla}$, has for kernel the line generated by λ.

Then α is weakly absolute.

Proof. As the map μ is surjective, the component S_α of the Hodge locus determined by α is smooth with tangent space K_λ at $0 \in T$ (cf [10], Proposition 5.14).

The conditions 2) and 3) imply that any constant sub-variation of Hodge structure of $R^{2k}\pi_*\mathbb{Q}_{prim}$ defined along an open set of S_α containing the point 0 parameterizing X is equal to $\mathbb{Q}\alpha$. Indeed, if $\gamma^{p,q}$ is a locally constant section of $R^{2k}\pi_*\mathbb{C}_{prim}$ which remains of type (p,q) near 0 on S_α, where we may assume $p \geq q$ by Hodge symmetry, then we have

$$\overline{\nabla}\gamma^{p,q} = 0 \,\text{in}\, H^{p-1,q+1} \otimes \Omega_{S_\alpha},$$

and in particular, we have at 0,

$$\overline{\nabla}\gamma^{p,q}(0) = 0 \,\mathrm{in}\, H^{p-1,q+1}(X_0)_{prim} \otimes K_\lambda^*.$$

Thus by assumptions 2) and 3), we conclude that $\gamma^{p,q} = 0$ for $p > k$ and $\gamma^{p,q}$ is proportional to λ for $p = k$.

We conclude then by applying Theorem 0.5, 1). ■

Remark 3.2 *The same reasoning shows that if we only assume 1) and 2) in Theorem 3.1, then the class α satisfies the conclusion of part 2 of Theorem 0.5. Thus in particular, if $\widetilde{S}^0_\alpha$ is the irreducible component of $\widetilde{S}_\alpha$ passing through α (it is unique and reduced because $\widetilde{S}_\alpha$ is now smooth at the point α), then $p(\widetilde{S}^0_\alpha)$ is defined over $\overline{\mathbb{Q}}$.*

It is interesting to note that condition 1) is a Zariski open condition on the class $\lambda \in H^{k,k}$ (non necessarily Hodge) and that conditions 2) and 3) are Zariski open in the set where 1) is sastified. One can even note that the complementary set where these conditions are not satisfied, is Zariski closed and defined over $\mathbb{Q}$ as are the bundles $\mathcal{H}^{p,q}$ and the map $\overline{\nabla}$.

Of course, even if we can show that the Zariski open set of F^kH^{2k} defined by the condition 1), 2), 3) above is non empty, it is not clear if there are any Hodge classes in it. This is the case however if our variation of Hodge structure has Hodge level 2, that is $h^{p,q} = 0$ for $p \geq k+2$. Indeed, in this case, we have the Green density criterion (cf [10], 5.3.4) which guarantees that if there is any $\lambda \in H^{k,k}(X_t)_{prim}$ satisfying property 1), then the set of rational Hodge classes are topologically dense in the real part of the vector bundle $H^{k,k}$.

Example 3.3 *The criterion above allows us to prove that many Hodge classes are weakly absolute for surfaces in $\mathbb{P}^3$, without using the Lefschetz theorem on $(1,1)$-classes.*

More interestingly, it allows to show a similar result for certain level 2 subvariations of Hodge structure in the H^{2k} of a variety X, without knowing the Hodge-Grothendieck generalized conjecture for this sub-Hodge structure. We can construct such examples on 4-dimensional hypersurfaces with automorphisms.

Example 3.4 *Consider the action of the involution ι on $\mathbb{P}^5$ given by $\iota(X_0, \ldots, X_5) = (-X_0, -X_1, X_2, \ldots, X_5)$, and take for T the family of isomorphism classes of degree 6 hypersurfaces whose defining equation is invariant under ι, and for sub-Hodge structure the anti-invariant part of $H^4(X)$ under ι. This Hodge structure has Hodge level 2, because ι acts trivially on the rank 1 space $H^{4,0}(X)$. Thus the Green density criterion applies once assumption 1) above is satisfied. The parameter space T has dimension 226, and the number $h^{1,3}_-$ is equal to 208. One can check that assumptions 1), 2), 3) are satisfied generically on $F^2H^4_-$, thus proving that many Hodge classes are weakly absolute, even if the Hodge conjecture is not known for them. This is done following [3], [8] by computations in the Jacobian ring of the generic hypersurface as above. In fact this can be done for X the Fermat hypersurface, and for a generic class $\lambda \in H^{2,2}(X)_-$.*

We now turn to another application of Theorem 0.5, which concerns the fibers of the period map and the Torelli problem.

Let $\pi : \mathcal{X} \to T$ be a family of smooth projective varieties which is defined over $\mathbb{Q}$, and consider the variation of Hodge structure on $H^n(X_t)_{prim}$. Here T is assumed to be smooth. The corresponding infinitesimal variation of Hodge structure at $t \in T$ is given by the map

$$\overline{\nabla} : H^{p,q}(X_t)_{prim} \to H^{p-1,q+1}(X_t)_{prim} \otimes \Omega_{T,t}.$$

We will assume the following property : at the generic point $0 \in T$, the corresponding map

$$\mu : H^{p,q}(X_0)_{prim} \otimes T_{T,0} \to H^{p-1,q+1}(X_0)_{prim},$$
$$\eta \otimes v \mapsto \overline{\nabla}_v(\eta),$$

is surjective whenever $H^{p,q}(X_0)_{prim} \neq 0$. (This property is satisfied for example by the families of hypersurfaces or complete intersections in projective space.) We then have :

Theorem 3.5 *Let $Z \subset T \times T$ be the set of points (t, t') such that there exists an isomorphism of Hodge structures between $H^n(X_t, \mathbb{Q})_{prim}$ and $H^n(X_{t'}, \mathbb{Q})_{prim}$. Let $W \subset T \times T$ be (the underlying reduced scheme of) an irreductible component of Z which dominates T. Then under the assumptions above, W is defined over $\overline{\mathbb{Q}}$ and any Galois transform of W is again (the underlying reduced scheme of) an irreducible component of Z.*

Proof. We apply Theorem 0.5, 2). The set W above is Γ_{red} for an irreducible component Γ of the Hodge locus corresponding to the induced variation of Hodge structure of weight 0 on $H^n(X_t, \mathbb{Q})^*_{prim} \otimes H^n(X_{t'}, \mathbb{Q})_{prim}$ on $T \times T$. What we have to prove in order to apply Theorem 0.5,2) is the fact that if W dominates T by the first (or equivalently second) projection, then any constant sub-Hodge structure of

$$H^n(X_t, \mathbb{Q})^*_{prim} \otimes H^n(X_{t'}, \mathbb{Q})_{prim},\ (t, t') \in W,$$

must be of type $(0, 0)$.

By definition, for $(t, t') \in W$, the Hodge structures on

$$H^n(X_t, \mathbb{Q})_{prim},\ \ H^n(X_{t'}, \mathbb{Q})_{prim},$$

are isomorphic. Thus the Hodge structures on

$$H^n(X_t, \mathbb{Q})^*_{prim} \otimes H^n(X_{t'}, \mathbb{Q})_{prim},\ \ H^n(X_t, \mathbb{Q})^*_{prim} \otimes H^n(X_t, \mathbb{Q})_{prim}$$

are isomorphic. Furthermore, t is generic. Thus it suffices to prove that on any finite cover of T, there is no constant sub-Hodge structure of $H^n(X_t, \mathbb{Q})^*_{prim} \otimes H^n(X_t, \mathbb{Q})_{prim}$ which is not of type $(0, 0)$.

This is done by an easy infinitesimal argument. Let

$$\alpha \in H^n(X_t, \mathbb{Q})^*_{prim} \otimes H^n(X_t, \mathbb{Q})_{prim}$$

be of bidegree (r, s) with $r > s$, $r + s = 0$. Thus $r > 0$, and if we see α as an element of $Hom\,(H^n(X_t)_{prim}, H^n(X_t)_{prim})$, $\alpha \in H^{r,-r}$ means that $\alpha(H^{p,q}(X_t)_{prim}) \subset H^{p+r,q-r}(X_t)_{prim}$.

We have to show that if there is a flat section $\tilde{\alpha}$ on T, extending α and staying of type $(r, -r)$, then $\alpha = 0$. It suffices to show this at first order at $0 \in T$, where this is equivalent to say that if

$$\overline{\nabla}\alpha = 0 \in H^{r-1,-r+1}(X_0 \times X_0) \otimes \Omega_{T,0},$$

then $\alpha = 0$. But to say that $\overline{\nabla}\alpha = 0$ is equivalent to say that

$$\overline{\nabla}_v(\alpha(\phi)) = \alpha(\overline{\nabla}_v(\phi)),\ \forall \phi \in H^{p,q}(X_0)_{prim},\ \forall (p,q),\ p+q=n,\ \forall v \in T_{T,0}. \quad (3.1)$$

Equation (3.1) shows that α is in fact determined by its value on the first non 0 term $H^{p,q}(X_0)_{prim}$, because by assumption the map

$$H^{p,q}(X_0)_{prim} \otimes T_{T,0} \to H^{p-1,q+1}(X_0)_{prim},$$

$$\phi \otimes v \mapsto \overline{\nabla}_v(\phi)$$

is surjective once $H^{p,q}(X_0)_{prim}$ is different from 0.

On the other hand, α must be zero on the first non 0 term $H^{p,q}(X_0)_{prim}$, because it sends it in $H^{p+r,q-r}(X_0)_{prim} = 0$. ■

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